



NUMERICAL METHOD FOR THE SOLUTION OF MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

The numerical solution of a multi-order fractional differential equation with initial conditions is presented in this paper using a collocation approach in the sense of Caputo. After obtaining the problem's integral form, we converted it into a system of linear algebraic equations. To solve the algebraic problem and replace the exact solution, matrix inversion is used. A numerical example is given to illustrate the method's effectiveness and precision.

Keywords: Collocation, Differential Equation, Fractional derivatives, Approximate Solution.

Introduction

The fields of mathematics, physics, chemistry and engineering all heavily rely on fractional differential and integral equations. Ordinary and partial differential equations are examples of functional equations where real-world problems are mathematically modelled. Integro-differential equations are used in the modeling of some physical phenomena in science and engineering. They are present in many mathematical formulations of physical phenomena (IDEs). A few examples are the kinetic equations that form the basis of the kinetic theory of rarefied gases, plasma, radiation transfer and coagulation[1]. Some of the numerical solution of fractional differential equations developed in the literature include: Perturbed collocation method [2] Adomian decompositions method by[3-5], Collocation method by[6-9], Hybrid linear multistep method[10-11], Differential transform method[12], Pseudospectral Method[13], Bernstein Polynomials Method[14-15], [16] utilized a numerical approach based on the Boubaker polynomial to generate approximate numerical solutions to the multi-order fractional differential equations. Their decision was to use an operational matrix for fractional integration based on Boubakar polynomials.

In this work, we adopt collocation method for solving multi-order fractional differential equations of the form

$$D^{\beta}y(x) = \sum_{j=0}^N q_j D^{\alpha_j}y(x) + h(x) \quad (1.1)$$

subject to the initial condition

$$y^{(j)}(\alpha_j) = \lambda_j \quad j = 0, 1, \dots, n-1, n \in \mathbb{N} \quad \beta > \alpha_N \quad (1.2)$$

where $y(x)$ is the unknown function, D^{α_j} and D^{β} are the Caputo's derivative, $h(x)$ is the force known -prior. $q_j(x)$ is the known function, a_j and λ_j are known constants.

Basic Definitions

In this section, we present certain definitions and fundamental ideas of fractional calculus for the purpose of the formulation of the problem that has been presented.

Definition 1: The Caputo derivative with order $\alpha > 0$ of the given function $f(x)$, $x \in (a, b)$ is defined as

$${}_x^C D_a^\alpha y(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-s)^{m-\alpha-1} y^{(m)}(s) ds \quad (2.1)$$

where $m-1 \leq \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$

Definition 2: Let (a_n) , $n \leq 0$ be a sequence of real numbers. The power series in x with coefficients a_n is an expression.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \phi(x) A \quad (2.2)$$

where $\phi(x) = [1 \ x \ x^2 \ \cdots \ x^N]$, $A = [a_0 \ a_1 \ \cdots \ a_N]^T$
then $y(x, n) = x^n A$, $n = 0(1)N$, $n \in \mathbb{Z}^+$

Definition 3: The desired collocation points within an interval are determined using this method. i.e. $[a, b]$ and is provided by

$$x_i = a + \frac{(b-a)i}{N}, i = 0, 1, 2 \dots \dots N \quad (2.3)$$

Definition 4: Let $y(x)$ be a continuous function, then

$${}_0 I_x^\beta ({}_x^C D_a^\alpha y(x)) = y(x) - \sum_{k=0}^N \frac{y^{(k)}(0)}{k!} x^k \quad (2.4)$$

where $m-1 < \beta < 1$

Definition 5: Let $p(s)$ be an integrable function, then

$${}_0 I_x^\beta (p(s)) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} p(s) ds \quad (2.5)$$

Methodology

In this section, we develop a collocation method that uses power series polynomials as our fundamental function to solve multi-order fractional differential equations numerically.

Let $y(x)$ be a solution to (1.1) subject to (1.2), the integral form is

$$y(x) = W(x) + \sum_{j=0}^N \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} q_j(s) \left[\int_0^s (s-t)^{m-\alpha-1} y^{(m)}(t) dt \right] ds \quad (3.0)$$

where

$$W(x) = \sum_{k=0}^N \frac{y^{(k)}(0)}{k!} x^k \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s) ds$$

multiply equation (1.1) by ${}_0 I_x^\beta (\cdot)$ gives

$${}_0 I_x^\beta (D^\beta y(x)) = {}_0 I_x^\beta (\sum_{j=0}^N q_j D^{\alpha_j} y(x)) + {}_0 I_x^\beta (h(x)) \quad (3.1)$$

using equation (2.4) on equation (3.1) gives

$$y(x) = \sum_{k=0}^N \frac{y^{(k)}(0)}{k!} x^k + {}_0 I_x^\beta (\sum_{j=0}^N q_j D^{\alpha_j} y(x)) + {}_0 I_x^\beta (h(x)) \quad (3.2)$$

applying equation (2.1) and (2.5) to equation (3.2) gives

$$y(x) = \sum_{k=0}^N \frac{y^{(k)}(0)}{k!} x^k +$$

$$\frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} [\sum_{j=0}^N q_j(x) \frac{1}{\Gamma(m_j-\alpha_j)} \int_0^s (s-t)^{m-\alpha-1} y^{(m_j)}(t) dt] ds \quad (3.3)$$

substituting equation (2.2) into equation (3.3) gives

$$y(x) = \sum_{k=0}^N \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} [\sum_{j=0}^N q_j(x) \frac{1}{\Gamma(m_j-\alpha_j)} \int_0^s (s-t)^{m-\alpha-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t)) dt] ds \mathbf{A} \quad (3.4)$$

collocating at x_i in equation (3.4) gives

$$y(x_i) = W(x_i) + \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} q_j(s) [\sum_{j=0}^N \frac{1}{\Gamma(m_j-\alpha_j)} \int_0^s (s-t)^{m-\alpha-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t)) dt] ds \mathbf{A} \quad (3.5)$$

where

$$W(x_i) = \sum_{k=0}^N \frac{y^{(k)}(0)}{k!} x^k \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} h(s) ds$$

simplify equation (3.5) gives

$$\phi(x_i) \mathbf{A} = W(x_i) + \left[\frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} q_j(s) \sum_{j=0}^N \frac{1}{\Gamma(m_j-\alpha_j)} \int_0^s (s-t)^{m-\alpha-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t)) dt ds \right] \mathbf{A} \quad (3.6)$$

factorize the values of A from equation (3.6) gives

$$\left[\frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} q_j(s) \sum_{j=0}^N \frac{1}{\Gamma(m_j-\alpha_j)} \int_0^s (s-t)^{m-\alpha-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t)) dt ds \right] \mathbf{A} = W(x_i) \quad (3.7)$$

equation (3.7) can be in the form

$$V(x_i) \mathbf{A} = W(x_i) \quad (3.8)$$

where

$$V(x_i) = \left[\frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i-s)^{\beta-1} q_j(s) \sum_{j=0}^N \frac{1}{\Gamma(m_j-\alpha_j)} \int_0^s (s-t)^{m-\alpha-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t)) dt ds \right] \quad (3.9)$$

and

$$\mathbf{A} = [a_0 \ a_1 \ \dots \ a_N]^T$$

multiply both sides of equation (3.8) by $V^{-1}(x_i)$ gives

$$\mathbf{A} = V^{-1}(x_i) W(x_i) \quad (3.10)$$

substituting (3.10) in the approximate solution gives

$$y(x) = \phi(x_i) V^{-1}(x_i) W(x_i) \quad (3.11)$$

Numerical Examples

In this section, we considered numerical example to evaluate the effectiveness and clarity of the method. A MAPLE 18 program is used to perform the computations. Let $y_n(x)$ and $y(x)$ be the approximate and exact solutions respectively. $Error_N = |y_n(x) - y(x)|$.

Example: [2] Consider multi-order Fractional differential equation .

$$D^{1.5} y(x) = -x^{-1} D^{0.5} y(x) - x^{0.5} y(x) + f(x)$$

with this condition $y'(0) = y(0) = 0$ and exact solution $y(x) = x^3 - x^2$



Solution: We solve for $N=4$

$$y_4 = 1.365574320288940 \times 10^{-14} - 2.546407529280260 \times 10^{-12}x - 0.99999999275360x^2 + 0.999999998413614x^3 + 6.644427230639850 \times 10^{-10}x^4$$

Table 1: Exact, approximate and absolute error values for Example

x	Exact	Our method_{N=4}	error₄	error_{[2]=4}
0.1	-0.900000000e-2	-0.8999995000e-2	5.0000000e-12	2.6668e-10
0.2	-0.32000000e-1	-0.3199980000e-1	2.0000000e-11	9.6994e-10
0.4	-0.96000000e-1	-0.9599980000e-1	2.0000000e-11	3.0781e-09
0.6	-0.1440000000	-0.14399999990	1.0000000e-10	5.0056e-09
0.8	-0.128000000	-0.128000000100	1.0000000e-10	5.2733e-09
1.0	0.0000000000	-2.35557276e-10	2.3572769e-10	2.2421e-09

Conclusion

This work develops a numerical method for solving initial condition containing fractional differential equations using collocation. The used approach is reliable, effective, and simple to compute. The approximate solution at $N = 4$ gives what we observed from the result obtained as shown in Table1. $y_4(x) = 1.365574320288940 \times 10^{-14} - 2.546407529280260 \times 10^{-12}x + 1.000000001121990x^2 - 1.000000001716120x^3 + 6.387779194483300 \times 10^{-10}x^4$. The numerical result almost converge to exact solution and give very lesser errors. We also compare our absolute errors as shown in Table 1, this clearly shows that our method perform better.

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